# Transition Maths and Algebra with Geometry

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Lecture Notes
Electrical and Computer Engineering

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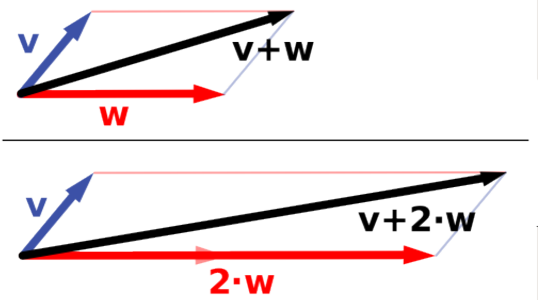
# Contents

# Well known vector spaces

You are familar with the 2D vector space:

$$\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}$$

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## $\mathbb{R}^2$ vector space

In $\mathbb{R}^2$ vectors can be added and multiplied by a number called a *scalar*:

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, y_1 + y_2),$$

$$\lambda \odot (x, y) = (\lambda \cdot x, \lambda \cdot y) \text{ for } \lambda \in \mathbb{R}$$

# Definition

## Definition

Let $\mathbb{K}$ be a field. A set $\mathbf{V}$ together with two operations:

- vector addition $+ : \mathbf{V} \times \mathbf{V} \to \mathbf{V}$ and
- scalar multiplication $\cdot : \mathbb{K} \times \mathbf{V} \to \mathbf{V}$

is called a *vector space* over the field $\mathbb{K}$ if it satisfies the following conditions.

- for any $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{V}$ we have $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ and $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$,
- there is $\mathbf{0} \in \mathbf{V}$ called the *zero vector* such that for any $\mathbf{v} \in \mathbf{V}$ we have $\mathbf{v} + \mathbf{0} = \mathbf{0} + \mathbf{v} = \mathbf{v}$,
- for any $\mathbf{v} \in \mathbf{V}$ there is $-\mathbf{v} \in \mathbf{V}$ such that $\mathbf{v} + -\mathbf{v} = -\mathbf{v} + \mathbf{v} = \mathbf{0}$,

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# Definition

- for any $\mathbf{u}, \mathbf{v} \in \mathbf{V}$ and any scalar $\lambda \in \mathbb{K}$ we have $\lambda \cdot (\mathbf{u} + \mathbf{v}) = \lambda \cdot \mathbf{u} + \lambda \cdot \mathbf{v}$
- for any $\mathbf{u} \in \mathbf{V}$ and any scalar $\lambda_1, \lambda_2 \in \mathbb{K}$ we have $(\lambda_1 + \lambda_2) \cdot \mathbf{u} = \lambda_1 \cdot \mathbf{u} + \lambda_2 \cdot \mathbf{u}$
- for any $\mathbf{u} \in \mathbf{V}$ and any scalars $\lambda_1, \lambda_2 \in \mathbb{K}$ we have $(\lambda_1 \cdot \lambda_2) \cdot \mathbf{u} = \lambda_1 \cdot (\lambda_2 \cdot \mathbf{u})$
- for any $\mathbf{u} \in \mathbf{V}$ we have $1 \cdot \mathbf{u} = \mathbf{u}$.

The elements of $\mathbf{V}$ are called *vectors* and the elements of $\mathbb{K}$ are *scalars*.

# Examples

Let $\mathbb{K}$ be any field and let $n \in \mathbb{N}$ be a natural number. Then

$$\mathbb{K}^n = \{(x_1, \ldots, x_n) \mid x_i \in \mathbb{K}\}$$

together with vector addition

$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n)$$

and scalar multiplication

$$\lambda \cdot (x_1, \ldots, x_n) = (\lambda \cdot x_1, \ldots, \lambda \cdot x_n) \text{ for any } \lambda \in \mathbb{K}$$

forms a vector space over $\mathbb{K}$. The zero vector $\mathbf{0}$ is in this case given by

$$\mathbf{0} = (0, \ldots, 0)$$

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# Examples

In the case when $\mathbb{K} = \mathbb{R}$ and $n = 2$ we get the well known space $\mathbb{R}^2$.

If $\mathbb{K} = \mathbb{C}$ we obtain new space

$$\mathbb{C}^n = \{(z_1, \ldots, z_n) \mid z_i \in \mathbb{C}\}.$$

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# Complex numbers as a vector space

Recall that the set of complex numbers $\mathbb{C}$ has been defined as

$$\mathbb{C} = \mathbb{R} \times \mathbb{R} = \{(a, b) \mid a, b \in \mathbb{R}\}.$$

The set $\mathbb{C}$ together with the standard complex numbers addition can be view as a vector space over the field $\mathbb{R}$.

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# Non-standard examples

Let $\mathbb{K}$ be a field. The set $\mathbb{K}[x]$ of all polynomials over $\mathbb{K}$ with the standard polynomial addition and multiplication by a constant from $\mathbb{K}$ forms a vector space over $\mathbb{K}$. The zero vector $\mathbf{0}$ is the zero polynomial 0. Similarly, the set $\mathbb{K}_n[x]$ of polynomials of degree less than or equal to $n$ over the field $\mathbb{K}$ is a vector space.

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# Definition

## Definition

Let $\mathbf{V}$ be a vector space over $\mathbb{K}$. A subset $\mathbf{W} \subseteq \mathbf{V}$ is called a *subspace* of $\mathbf{V}$ if $\mathbf{W}$ is a vector space over $\mathbb{K}$ under the same operations of vector addition and scalar multiplication.

To put it differently, a subset $\mathbf{W} \subseteq \mathbf{V}$ is a subspace of $\mathbf{V}$ if

- for any $\mathbf{w_1}, \mathbf{w_2} \in \mathbf{W}$ we have $\mathbf{w_1} + \mathbf{w_2} \in \mathbf{W}$
- $\mathbf{0} \in \mathbf{W}$
- for any $\mathbf{w} \in \mathbf{W}$ we have $-\mathbf{w} \in \mathbf{W}$
- for any scalar $\lambda \in \mathbb{K}$ and any $\mathbf{w} \in \mathbf{W}$ we have $\lambda \cdot \mathbf{w} \in \mathbf{W}$

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# Examples

Consider the vector space $\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$ with the standard vector addition and scalar multiplication. Define the set $W$ by

$$W = \{(x, y, z) \mid 2x + y + 3z = 0\}$$

$W$ is a subspace of $\mathbb{R}^3$. On the other hand the set $W'$

$$W' = \{(x, y, z) \mid x^2 = 1\}$$

is *not* a subspace of $\mathbb{R}^3$.

# Examples

Consider the vector space $\mathbb{C}$ over $\mathbb{R}$. The set

$$\{z \in \mathbb{C} \mid \mathfrak{Re}(z) = 0\}$$

is a subspace of $\mathbb{C}$.

# Linear combination

## Definition

Let $\mathbf{V}$ be a vector space over a field $\mathbb{K}$ and let $a_1, \ldots, a_n \in \mathbb{K}$ and $\mathbf{v_1}, \ldots, \mathbf{v_n} \in \mathbf{V}$. The vector

$$a_1\mathbf{v_1} + \ldots + a_n\mathbf{v_n} \in \mathbf{V}$$

is called the *linear combination* of the vectors $\mathbf{v_1}, \ldots, \mathbf{v_n}$ with coefficients $a_1, \ldots, a_n$.

Example: In $\mathbb{R}^2$ the vector $(2, 3)$ is a linear combination of $(1, 1)$ and $(0, 1)$ with coefficients 2 and 1:

$$(2, 3) = 2 \cdot (1, 1) + 1 \cdot (0, 1)$$

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# Span

## Definition

Let $\mathbf{V}$ be a vector space over a field $\mathbb{K}$ and let $S \subseteq \mathbf{V}$ be a set of vectors. By the *span* of the set $S$ of vectors we mean the set $span(S) \subseteq \mathbf{V}$ defined by

$$span(S) = \{a_1\mathbf{v_1} + \ldots + a_n\mathbf{v_n} \mid a_i \in \mathbb{K} \text{ and } \mathbf{v}_i \in \mathbf{S}\}.$$

In other words, $span(S)$ is the set of all possible linear combinations of vectors from $S$.

Example: Consider $\mathbb{R}^3$ and two vectors $(1,2,0),(0,0,3) \in \mathbb{R}^3$.

$$span(\{(1,2,0),(0,0,3)\}) = \{a(1,2,0) + b(0,0,3) \mid a, b \in \mathbb{R}\} = \{(a,2a,3b) \mid a,b \in \mathbb{R}\}$$

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# Span

**Theorem**

Let $\mathbf{V}$ be a vector space over a field $\mathbb{K}$ and let $\mathbf{v_1}, \ldots, \mathbf{v_n} \in \mathbf{V}$. Then the set

$$span(\{\mathbf{v_1}, \ldots, \mathbf{v_n}\}) = \{a_1\mathbf{v_1} + \ldots + a_n\mathbf{v_n} \mid a_i \in \mathbb{K}\}.$$

is a subspace of $\mathbf{V}$ containing the vectors $\mathbf{v_1}, \ldots, \mathbf{v_n}$.

Proof: Take $\mathbf{u}, \mathbf{w} \in span(\{\mathbf{v_1}, \ldots, \mathbf{v_n}\})$. This means that

$$\mathbf{u} = a_1\mathbf{v_1} + \ldots + a_n\mathbf{v_n},$$
$$\mathbf{w} = b_1\mathbf{v_1} + \ldots + b_n\mathbf{v_n},$$

for $a_i, b_i \in \mathbf{K}$.

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# Span

- is $\mathbf{u} + \mathbf{w} \in span(\{\mathbf{v_1}, \ldots, \mathbf{v_n}\})$ ? Yes, because

$$\mathbf{u} + \mathbf{w} = a_1\mathbf{v_1} + \ldots + a_n\mathbf{v_n} + b_1\mathbf{v_1} + \ldots + b_n\mathbf{v_n} =$$
$$(a_1 + b_1)\mathbf{v_1} + \ldots + (a_n + b_n)\mathbf{v_n}$$

- is $-\mathbf{u} \in span(\{\mathbf{v_1}, \ldots, \mathbf{v_n}\})$ ? Yes, because

$$-\mathbf{u} = -(a_1\mathbf{v_1} + \ldots + a_n\mathbf{v_n}) = (-a_1)\mathbf{v_1} + \ldots + (-a_n)\mathbf{v_n}$$

- is $\mathbf{0} \in span(\{\mathbf{v_1}, \ldots, \mathbf{v_n}\})$? Yes, because

$$\mathbf{0} = 0\mathbf{v_1} + \ldots + 0\mathbf{v_n},$$

- is $\lambda \cdot \mathbf{u} \in span(\{\mathbf{v_1}, \ldots, \mathbf{v_n}\})$ for any $\lambda \in \mathbb{K}$? Yes, because

$$\lambda \cdot \mathbf{u} = \lambda(a_1\mathbf{v_1} + \ldots + a_n\mathbf{v_n}) = (\lambda a_1)\mathbf{v_1} + \ldots + (\lambda a_n)\mathbf{v_n}.$$

## Span: Examples

- $\mathbb{C}$ over $\mathbb{R}$.

$$span(\{1,i\}) = \{a+bi \mid a,b \in \mathbb{R}\} = \mathbb{C},$$
$$span(\{i,2+i\}) = \mathbb{C},$$
$$span(\{i+3\}) = \{3a+ai \mid a \in \mathbb{R}\} \neq \mathbb{C}.$$

- $\mathbb{R}[x]$ over $\mathbb{R}$

$$span(\{x^2,x,1\}) = span(\{x^2,x,1,x^2+5\}) =$$
$$\{ax^2+bx+c \mid a,b,c \in \mathbb{R}\} \text{ -quadratic functions.}$$

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# Linear independence

## Definition

Let $\mathbf{V}$ be a vector space over a field $\mathbb{K}$. We say that a set $S \subseteq \mathbf{V}$ of vectors from $\mathbf{V}$ is *linearly independent* if for any $\mathbf{v_1}, \ldots, \mathbf{v_n} \in S$ and any $a_1, \ldots, a_n \in \mathbb{K}$

$$a_1\mathbf{v_1} + \ldots + a_n\mathbf{v_n} = \mathbf{0} \implies a_1 = \ldots = a_n = 0.$$

Otherwise, the set $S$ is *linearly dependent*.

Example: $\mathbb{R}^2$ over $\mathbb{R}$: $\{(0,1),(1,1)\}$ is linearly independent because the linear combination

$$a(0,1) + b(1,1) = (a, a+b)$$

is equal the zero vector $(0,0)$ if $a = 0$ and $a + b = 0$. This implies that $a = b = 0$.

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# Linear Independence: Examples

- $\mathbb{C}$ over $\mathbb{R}$: $\{1, i\}$ is linearly independent because

$$a \cdot 1 + b \cdot i = 0 \implies a = b = 0.$$

- $\mathbb{R}[x]$ over $\mathbb{R}$: $\{x+1, x-1, 2\}$ is linearly dependent because

$$(x+1) + (-1)(x-1) + (-1)2 = 0.$$

- $\mathbb{R}[x]$ over $\mathbb{R}$: Is $\{1, x, x^2, x^3, \ldots\}$ a linearly independent set?

# Linear independence: properties

## Fact

Let $\mathbf{V}$ be a vector space over $\mathbb{K}$. The set $\{\mathbf{v_1}, \ldots, \mathbf{v_n}\} \subseteq \mathbf{V}$ is linearly dependent if and only if at least one $\mathbf{v_i}$ of the vectors from $\{\mathbf{v_1}, \ldots, \mathbf{v_n}\}$ can be expressed as the linear combination of others.

## Fact

Let $\mathbf{V}$ be a vector space over $\mathbb{K}$. If the set $S \subseteq \mathbf{V}$ is linearly independent then any subset $S' \subseteq S$ of the set $S$ is also linearly independent.

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# Basis

## Definition

Let $\mathbf{V}$ be a vector space over a field $\mathbb{K}$. A set $B \subseteq \mathbf{V}$ of vectors is called *a basis of* $\mathbf{V}$ if

- $B$ is linearly independent,
- $span(B) = \mathbf{V}$.

For the vector space $\mathbb{R}^2$ over $\mathbb{R}$: $\{(1,0),(1,1)\}$ is a basis of $\mathbb{R}^2$.

# Basis: Examples

- $\mathbb{C}$ over $\mathbb{R}$: The set $\{1, i\}$ is a basis of $\mathbb{C}$. So is e.g. $\{1, i+2\}$.
- Let $\mathbb{K}$ be a field. For the vector space $\mathbb{K}^n$ over $\mathbb{K}$ the set

$$\{(1,0,\dots,0),(0,1,\dots,0),\dots,(0,0,\dots,1)\}$$

  forms a basis of $\mathbb{K}^n$.
- $\mathbb{K}_n[x]$ over $\mathbb{K}$: The set

$$\{1, x, x^2, x^3, \dots, x^n\}$$

  is a basis of $\mathbb{K}_n[x]$.

# Basis: Properties

**Theorem**

Let $\mathbf{V}$ be a vector space over $\mathbb{K}$. A set $\{\mathbf{v_1}, \ldots, \mathbf{v_n}\} \subseteq \mathbf{V}$ is a basis of $\mathbf{V}$ if and only if any vector $\mathbf{w} \in \mathbf{V}$ can be uniquely expressed as a linear combination of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

Proof: ($\Rightarrow$) Let $\mathbf{w} = a_1\mathbf{v_1} + \ldots + a_n\mathbf{v_n}$ and $\mathbf{w} = b_1\mathbf{v_1} + \ldots + b_n\mathbf{v_n}$. This means that

$$a_1\mathbf{v_1} + \ldots + a_n\mathbf{v_n} = b_1\mathbf{v_1} + \ldots + b_n\mathbf{v_n}$$

Hence,

$$(a_1 - b_1)\mathbf{v_1} + \ldots + (a_n - b_n)\mathbf{v_n} = \mathbf{0}$$

Since $\{\mathbf{v_1}, \ldots, \mathbf{v_n}\}$ is linearly independent we have $a_1 - b_1 = 0, \ldots, a_n - b_n = 0$. Thus $a_1 = b_1, \ldots, a_n = b_n$.

# Basis: Properties

Proof: ($\Leftarrow$) Since any vector $\mathbf{w} \in \mathbf{V}$ may be expressed as the linear combination of $\{\mathbf{v_1}, \ldots, \mathbf{v_n}\}$ this means that $span(\{\mathbf{v_1}, \ldots, \mathbf{v_n}\}) = \mathbf{V}$. Was is left to be shown is linear independence of $\{\mathbf{v_1}, \ldots, \mathbf{v_n}\}$. Consider the zero vector $\mathbf{0}$. It can be expressed as the linear combination

$$\mathbf{0} = 0\mathbf{v_1} + \ldots + 0\mathbf{v_n}.$$

Since the above representation is unique for any $a_1, \ldots, a_n \in \mathbb{K}$

$$a_1\mathbf{v_1} + \ldots + a_n\mathbf{v_n} = \mathbf{0} \implies a_1 = a_2 = \ldots = a_n = 0.$$

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# Dimension

**Theorem**

Let a vector space $\mathbf{V}$ have a finite basis. Then any two bases of $\mathbf{V}$ are of the same size.

**Definition**

If a vector space $\mathbf{V}$ has a finite basis then the *dimension* of $\mathbf{V}$ is the size of any basis of $\mathbf{V}$. Otherwise, the *dimension* is defined to be $\infty$. We denote this number by $\dim(\mathbf{V})$.

Example: $\dim(\mathbb{R}^3) = 3$.

# Dimension: Examples

- $\dim(\{\mathbf{0}\}) = 0$,
- $\dim(\mathbb{K}^n) = n$,
- $\dim(\mathbb{K}[x]) = \infty$,
- $\dim(\mathbb{K}_n[x]) = n + 1$,
- $\dim(\mathbb{C} \text{ over } \mathbb{R}) = 2$,
- $\dim(\mathbb{C} \text{ over } \mathbb{C}) = 1$.

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# Dimension: Properties

**Theorem**

If $\mathbf{W}$ is a subspace of a vector space $\mathbf{V}$ then

- $\dim(\mathbf{W}) \leq \dim(\mathbf{V})$,
- $\dim(\mathbf{W}) = \dim(\mathbf{V}) \implies \mathbf{W} = \mathbf{V}$ for a space $\mathbf{V}$ with a finite dimension.

# Dimension: Properties

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# Dimension: Properties

**Theorem**

Let $\mathbf{V}$ be a vector space and $\dim(\mathbf{V}) = n$. Finally, let $S \subseteq \mathbf{V}$. Then:

- if $span(S) = \mathbf{V}$ then $|S| \geq n$,
- if $S$ is linearly independent and $|S| = n$ then $S$ is a basis,
- if $span(S) = \mathbf{V}$ and $|S| = n$ then $S$ is a basis,
- if $|S| > n$ then $S$ is linearly dependent.